HURWITZ NUMBERS FOR REGULAR COVERINGS OF SURFACES BY SEAMED SURFACES AND CARDY-FROBENIUS ALGEBRAS OF FINITE GROUPS

A.V.ALEXEEVSKI AND S.M.NATANZON(*)

Dedicated to S.P.Novicov on the occasion of his 70th birthday

Abstract. Analogue of classical Hurwitz numbers is defined in the work for regular coverings of surfaces with marked points by seamed surfaces. Class of surfaces includes surfaces of any genus and orientability, with or without boundaries; coverings may have certain singularities over the boundary and marked points. Seamed surfaces introduced earlier are not actually surfaces. A simple example of seamed surface is book-like seamed surface: several rectangles glued by edges like sheets in a book.

We prove that Hurwitz numbers for a class of regular coverings with action of fixed finite group G on cover space such that stabilizers of generic points are conjugated to a fixed subgroup $K \subset G$ defines a new example of Klein Topological Field Theory (KTFT). It is known that KTFTs are in one-to-one correspondence with certain class of algebras, called in the work Cardy-Frobenius algebras. We constructed a wide class of Cardy-Frobenius algebras, including particularly all Hecke algebras for finite groups. Cardy-Frobenius algebras corresponding to regular coverings of surfaces by seamed surfaces are described in terms of group G and its subgroups. As a result, we give an algebraic formula for introduced Hurwitz numbers.

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1. Introduction

In the paper we construct new examples of Klein Topological Field Theories (hereinafter abbreviated to KTFT) defined in [1]. The examples are connected with Hurwitz numbers for regular coverings of Klein surfaces by seamed surfaces. These numbers, generalizing classical Hurwitz numbers, are introduced in the work.

A classical Hurwitz number is a weighted number of non-equivalent meromorphic maps $f: C \to P$ of Riemann surfaces C to a fixed Riemann surface P such that topological types of critical values $p_i \in P$ of $f(z)$ coincide with prescribed data. Initially these numbers were introduced by Hurwitz [10] for Riemann sphere P. A topological type of a critical value p_i is defined by a partition α_i of n, the degree of f, into the unordered sum of positive integers. Actually, Hurwitz numbers depend on a Riemann surface P up to isomorphism, a number of critical values and a set of partitions α_i assigned to points of P that are critical values of f. Clearly, the partitions of n are in one-to-one correspondence with conjugacy classes of the symmetric group S_n.

Forgetting of the analytic structure, Hurwitz numbers can be defined as weighted numbers of non-equivalent branched coverings of a fixed oriented topological surfaces P with prescribed types of branching at fixed set of marked points. Again, branching type is described by a conjugacy class of the symmetric group. Moreover, "analytical" and topological Hurwitz numbers coincide.

Hurwitz numbers can be generalized to the case of regular coverings of a surface P with marked points [8, 5]. Let a finite group G acts effectively and continuously on a surface C. Then the projection of C onto the orbit space C/G is a branching covering and the topological type of the branching at a singular point in C/G is described by a conjugacy class α of G. Let $r: C/G \to P$ be a homeomorphism such that images of singular points coincide with marked points of P. Then the composite map $C \to P$ is called a G-covering of P. Assigning a conjugacy class α_i to a marked point p_i, we can define a Hurwitz number in the same way as in classical case. If $G = S_n$ then Hurwitz numbers of G-coverings actually coincide with classical Hurwitz numbers.

In [1] Hurwitz numbers were extended to a wider category $\mathcal{K}$ of surfaces. An object of $\mathcal{K}$ is a compact topological surface of any genus with or without boundary, orientable or non-orientable and with fixed set of marked points both in the interior and on the boundary of the surface. A covering of a surface $\Omega \in \mathcal{K}$ is defined as a continuous map $f: T \to \Omega$ of a surface $T \in \mathcal{K}$ onto Ω satisfying the following conditions: f is a local homeomorphism at interior non-marked points; f is a branching covering in the neighborhood of an interior marked point; either f is a homeomorphism or f is topologically equivalent to the complex conjugation map $z \to \bar{z}$ in a neighborhood of a point $z \in T$ such that $f(z)$ is boundary non-marked point.

Topological type of a covering $f: T \to \Omega$ at boundary marked point is defined by an equivalence class β of ordered pairs (s_1, s_2) of two elements of order 1 or 2 of symmetric group S_n. The equivalence is defined by $(s_1, s_2) \sim (gs_1g^{-1}, gs_2g^{-1})$ for $g \in S_n$. Actually, Hurwitz numbers of [1] depend on a surface $\Omega \in \mathcal{K}$ up to isomorphism in category $\mathcal{K}$, a set of conjugacy classes α_i of group S_n assigned to interior marked points of Ω and a set β_j of equivalence classes of pairs (s_1, s_2) assigned to boundary marked points.

A real algebraic curve is a pair (P, τ), where P is a Riemann surface (representing the complexification of the real algebraic curve) and $\tau: P \to P$ is an antiholomorphic involution (representing the complex conjugation on the complexification). The complex structure on P generates on the factor space $\Omega = P/\tau$ the structure of a Klein surface [16].

Forgetting this structure, we obtain that Ω is a surface from the category $\mathcal{K}$. Conversely, each $\Omega \in \mathcal{K}$ can be endowed with a structure that comes from a real algebraic curve. By above, Hurwitz numbers of [1] can be considered as Hurwitz numbers for real meromorphic maps with values in a fixed real algebraic curve.

Next generalization of Hurwitz numbers were done in [2, 3] where coverings of surfaces $\Omega \in \mathcal{K}$ by so called seamed surfaces were considered. Combining approaches, in this work we deal with G-coverings of surfaces by seamed surfaces.

Seamed surfaces appear first in physical models [19, 11]. Let $\Omega_i \in \mathcal{K}$ be a set of surfaces with marked points. Boundary marked points divide boundary contours of surfaces into segments. Let a set of homeomorphisms of the segments is given. Then we can identify a point of a segment with all its images. Obtained topological space is called a (topological) seamed surface. Images of Ω_i in the seamed surface are called regular parts. Note, that more then two segments of surfaces Ω_i can be identified with one segment of the seamed surface. Thus, a seamed surface generally is not a surface. Several copies of a rectangle glued by edges like sheets in a book give an example of a seamed surface.

Let finite group G acts effectively and continuously on a seamed surface T. Suppose, G/T is a surface (not seamed surface, but a surfaces of category $\mathcal{K}$). Then the projection $f : T \to T/G$ is a local homeomorphism at a generic point z of T. Note that the stabilizer K of z is not necessarily trivial subgroup. Clearly, stabilizers of other generic points of T are conjugated to K. Fix one of that stabilizers, namely, K. The topological type of the covering f at interior singular point p is defined by a conjugacy class α of the group $N = N_G(K)/K$ where $N_G(K)$ denotes the normalizer of the subgroup. It is shown in section 2.3 that topological type of f at boundary singular point q is described by an equivalence class β of an ordered pair (S_1, S_2) of subgroups such that $S_1 \supset K$, $S_2 \supset K$. The equivalence is defined by formula $(S_1, S_2) \sim (gS_1g^{-1}, gS_2g^{-1})$ for $g \in N_G(K)$.

By analogy with the case of coverings of surfaces by surfaces in category $\mathcal{K}$ we define "topological" Hurwitz numbers for G-coverings of surfaces by seamed surfaces (see section 2.4).

Following [17, 18] it is possible to define structures of Klein surfaces on regular parts of topological seamed surfaces. (The case of orientable regular parts of a seamed surface and and Riemann structures on them was considered in [19].) Using these structures we can define "analytical" Hurwitz numbers. It follows from [17, 18] that "topological" and "analytical" Hurwitz numbers coincide. In this work we will explore the topological category of seamed surfaces.

Topological field theories were introduced in works [4, 20]. In the case of dimension two it is a system of tensors that are connected with closed (that is oriented without boundary) surfaces with marked points. Let P be a such surface and V be a fixed vector surface. Assign vector space $V_i \approx V$ to each marked point p_i of P and put $V_P = \otimes_i V_i$. Then the system of linear maps $\Phi_P : V_P \to \mathbb{C}$ is call a closed topological field theory if certain axioms are satisfied. It follows from [8] that the Hurwitz numbers for the G-coverings generate the tensors that form a closed topological field theory .

Closed topological field theories are in one-to-one correspondence with commutative Frobenius algebras equipped with a fixed nondegenerate symmetric bilinear form and an involutive antiautomorphism [7, 1]. For a finite group G this Frobenius algebra is isomorphic to the center A of the group algebra of G [8].

Closed topological field theories were generalized to the case of oriented surfaces with boundaries [13, 14]. These theories are called open-closed topological field theories. Klein topological field theories (KTFTs)[1] are generalizations of open-closed topological field

theories to nonorientable surfaces. To define a KTFT, let us fix two vector spaces, A and B. For any $\Omega \in \mathcal{K}$ we assign a vector space A_i isomorphic to A to each interior marked point p_i of Ω and a vector spaces B_j isomorphic to B to each boundary marked point q_j. KTFT is a system of linear maps (correlators) $\Phi_\Omega : V_\Omega \to \mathbb{C}$ where $V_\Omega = (\otimes_i A_i) \otimes (\otimes_j B_j)$, satisfying axioms (see section 3.2). Its restriction to subcategory of orientable surfaces is open-closed topological field theory.

Relations between correlators and their algebraic implementations were described in [13, 14] for open-closed topological field theory and in [1] for KTFT. It was shown in [1] that there is a one-to-one correspondence between KTFT and certain class of algebras. Algebras of that class was called structure algebras in [1] and renamed to Cardy-Frobenius algebras in [2, 3]. Open-closed topological field theories can be considered as restrictions of KTFTs to the oriented surfaces. The one-to-one correspondence between open-closed topological field theories and Frobenius algebras endowed with additional structures was proved also independently in [12, 15].

By definition, a Cardy-Frobenius algebras H is a sum $H = A \oplus B$ of a commutative Frobenius subalgebra A and (typically noncommutative) Frobenius algebra B which is a two-sided ideal of H. Algebra H is endowed with additional structures: unit elements in algebras A and B, fixed nondegenerate symmetric invariant bilinear forms on A and B, involutive antiautomorphisms of A and B, element $U \in A$. Main condition establishes a relation between A and B. That relation is an implementation of Cardy condition (see section 4.1).

Semisimple Cardy-Frobenius algebras were classified in [1].

In this paper we construct a series of new examples of KTFTs. First, we prove that Hurwitz numbers for G-coverings of surfaces by seamed surfaces with fixed finite group G and subgroup K which is the stabilizer of a generic point, generate a Cardy-Frobenius algebra. Hurwitz numbers of regular coverings in the case $G = S_n$ and $K = \{e\}$ coincides with Hurwitz numbers of coverings of Klien surfaces by seamed surfaces [2, 3]. The KTFT corresponding to the latter Hurwitz numbers and Cardy-Frobenius algebra corresponding to that KTFT were described in [2, 3]. Note, that noncommutative part B of this Cardy-Frobenius algebra is an algebra of bipartite graphs.

KTFTs corresponding to G-coverings of surfaces by seamed surfaces are included in the work in broader series of examples of KTFTs: we prove that a KTFT corresponds to any effective action of a finite group N on a finite set X. Noncommutative parts of Cardy-Frobenius algebras for those KTFTs include all Hecke algebras of finite groups. We are grateful to E.Vinberg who pointed to us this connection. All constructed Cardy-Frobenius algebras $H = A \oplus B$ are semisimple. To prove it we construct a canonic faithful representation of algebra H in vector space V_X of formal linear combinations of elements of X and prove that the image coincides with the set of intertwining operators for natural representation of the group N in V_X.

Following [1], we obtain finally the formula for Hurwitz numbers for G-coverings of surfaces by seamed surfaces in terms of Cardy-Frobenius algebra (see section 5).

In section 2 seamed surfaces and G-coverings of surfaces by seamed surfaces are defined. Local topological invariants of G-coverings are described. Using these invariants Hurwitz numbers are defined and their topological properties are described. In section 3 by means of Hurwitz numbers for G-coverings of surfaces by seamed surfaces we construct a Klein Topological Field Theory. In section 4 we recall a definition of a Cardy-Frobenius algebra in a bit more invariant form than in [1] and associate a Cardy-Frobenius algebra with any action of a finite group G on a finite set. In section 5 we show that the Cardy-Frobenius

algebra of G-coverings of surfaces by seamed surfaces is isomorphic to the Cardy-Frobenius algebra generated by the action of of group $N = N_G(K)/K$ on a certain finite set. Finally, we give an algebraic formula for the Hurwitz numbers.

2. Regular coverings of surfaces by seamed surfaces

2.1. **Surfaces and seamed surfaces.** Two-dimensional compact manifold Ω, not necessarily orientable or connected, with or without a boundary, and with finitely many marked points is called in the work *a surface* for short. Thus, the boundary $\partial\Omega$ of a surface Ω is either empty or consists of finite number of contours. Some of the marked points belong to the interior $\Omega \setminus \partial\Omega$ of Ω other belong to the boundary. We require that any boundary contour contains at least one marked point. An interior not marked point is called simple. Orientation of a small neighborhood of a point is called *a local orientation at the point.*

Using marked points we define a stratification of a surface Ω as follows. Marked points are considered as 0-dimensional strata (0-strata), 1-strata are defined as boundary segments between neighbor boundary marked points at the same boundary contour, and 2-strata are defined as $\Omega^c \setminus (\partial\Omega^c \cup \Omega^c_0)$ where Ω^c is a connected component of Ω and Ω^c_0 denotes the set of marked points in Ω^c.

A homeomorphism $\phi : \Omega' \to \Omega''$ of surfaces is called *an isomorphism* if the image of the set of marked points of Ω' coincides with the set of marked points of Ω''. Therefore, an isomorphism is compatible with the stratifications of surfaces.

Closed 1-stratum of a surface Ω can be either a segment or a circle, the latter case corresponds to a boundary contour with just one marked point. Let δ_1 and δ_2 be two distinct homeomorphic closed 1-strata of Ω. By a homeomorphism between δ_1 and δ_2 we may identify points of δ_1 and δ_2 and obtain new, glued topological surface.

We shell consider below stratified topological spaces obtained by gluing more than two closed 1-strata $\delta_1, \delta_2, \delta_3, \dots$ of a surface Ω into one stratum by means of homeomorphisms $\phi_{1,2} : \delta_1 \to \delta_2, \phi_{1,3} : \delta_1 \to \delta_3, \dots$. Several such gluings are allowed. The obtained topological space is called *a seamed surface* (compare with [19]). Seamed surfaces are 2-dimensional stratified topological spaces, i.e. all strata are homeomorphic to topological manifolds of dimension 2 or less. Thus, a seamed surface typically is not a surface.

Formal definition of a seamed surface is as follows. First, let us define its 1-dimensional analog, that is a topological graph. *A graph* Δ is a 1-dimensional stratified topological space with finitely many 0-strata (vertexes) and finitely many 1-strata (edges). It is required that the closure of any edge is homeomorphic either to a closed segment or to a circle, any vertex belongs to at least one edge, any edge is adjacent to one or two vertexes. Connectedness of Δ is not required. A graph such that any vertex belongs to at least two half-edges we call *an s-graph*. Clearly, the boundary $\partial\Omega$ of a surface Ω with marked points is an s-graph.

A morphisms of graphs $\varphi : \Delta' \to \Delta''$ is a continuous epimorphic maps of graphs compatible with the stratification, i.e. the restriction of φ to any open 1-stratum (interior of an edge) of Δ' is a local (therefore, global) homeomorphism with appropriate open 1-stratum of Δ''.

Let $(\Omega, \Delta, \varphi)$ be a triple consisting of a surface Ω (typically nonconnected), an s-graph Δ and a morphism of s-graphs $\varphi : \partial\Omega \to \Delta$. By identification points of $\partial\Omega$ with their images in Δ we obtain a 2-dimensional stratified topological space T. Clearly, we may and will identify Δ with the subset of T. 0-strata of T are vertexes of Δ and interior marked points of Ω, its 1-strata are edges of Δ, its 2-strata are 2-strata of Ω.

A 2-dimensional stratified topological space T homeomorphic (as a stratified surface) to a topological space obtained by the described procedure is called *a seamed surface.* Note, that any surface Ω (with marked points) is a seamed surface.

A topological space obtained from a number of copies of rectangles $R = \{z \in \mathbb{C} || \mathsf{Re}\, z| < 1, 0 \leq \mathsf{Im}\, z < 1\}$ by gluing segments $(-1, +1)$ of real axis into one segment (like sheets are glued in a book) we call *a book-like (open) seamed surface.*

Let Δ be a seamed graph. A connected component of $U \setminus v$ where U is appropriately small neighborhood of a vertex $v \in \Delta$ is contained in one edge; we call it a germ of the edge. Two germs are called equal if they are adjacent to the same vertex and have nonempty intersection. Thus, any edge has two germs. For seamed surfaces we can similarly define germs of 2-strata. We skip the details.

A pair $(v, g(e))$ consisting of a vertex v and a germ $g(e)$ of an edge adjacent to v is called *a flag.* Note, that if an edge e is a loop, i.e. both vertices of e coincide with v, then there are *two distinct flags* corresponding to the pair (v, e). If an edge e has two distinct vertexes then there is only one flag corresponding (v, e).

Let $C(X)$ be a topological cone over topological space X. Then the topological space $C(X) \setminus X$ is called *an open cone* over X.

Let T be a seamed surface, Δ be the seamed graph of T, Δ_b be the set of vertexes of Δ and U be an appropriate small neighborhood of a point $z \in T$. Then

- if $x \in (T \setminus \Delta)$ then U is homeomorphic to an open disc;
- if $x \in \Delta \setminus \Delta_b$ then U is homeomorphic to a book-like seamed surface;
- if $x \in \Delta_b$ then U is homeomorphic to an open cone over a certain connected graph Γ; vertexes of Γ are in one-to-one correspondence with flags of Δ adjacent to x and edges of Γ correspond to germs of 2-strata adjacent to x.

2.2. **Coverings and regular coverings.** Continuous epimorphic map $\varphi : T' \to T''$ of seamed surfaces T' and T'' such that the restriction of φ to any open stratum of T' is a local homeomorphism with a stratum of the same dimension of T'' we call *a covering* of seamed surfaces. A covering that is a homeomorphism is called *an isomorphism* of a seamed surface, an isomorphism of a seamed surface with itself is called *an automorphism.*

Let G be a finite group, T be a seamed surface and $G \to \mathsf{Aut}(T)$ be an action of G on T by automorphisms of seamed surface T. Typically, we assume that the action is effective. In that case we identify elements of G with their images. The seamed surface T is called a *G-seamed surface* (it would be better to write 'G-(seamed surface)'; nevertheless, we omit the brackets).

An isomorphism $\phi : T' \to T''$ of G-seamed surfaces is called a *G-isomorphism* if $g\phi(z) = \phi(gz)$ for any $g \in G$, $z \in T'$.

Let (T, Δ, ϕ) be a G-seamed surface. Then, obviously, the orbit space T/G has the structure of a seamed surface $(T/G, \Delta/G, \psi)$ where $\psi : \Delta/G \to T/G$ is the induced map of orbit space of G action on the seamed graph Δ to the orbit space T/G. Clearly, the map $r : T \to T/G$ is a covering of seamed surfaces. Below we restrict ourselves to the case of coverings of surfaces (defined in section 2.1) by seamed surfaces.

Fix a surface Ω with marked points and a finite group G. A covering $f : T \to \Omega$ of a surface by G-seamed surface is called *a G-covering* if there exists an isomorphism $f_* : T/G \to \Omega$ such that $f = f_* \circ r$ where $r : T \to T/G$ is the map of T onto orbit space T/G.

Let $f' : T' \to \Omega$ and $f'' : T'' \to \Omega$ be two G-coverings over the surface Ω. An isomorphism $F : T' \to T''$ is called an *isomorphism of G-coverings* if $f' = f'' \circ F$ and $F(g(x)) = gF(x)$ for any $x \in T'$ and $g \in G$. Isomorphic G-coverings over Ω are called *equivalent coverings*. Denote by $\mathsf{Aut}_G f$ the group of automorphisms of a G-covering f.

Two G-coverings $f' : T' \to \Omega'$ and $f'' : T'' \to \Omega''$ are called *topologically equivalent* if there exists a G-isomorphism of G-seamed surfaces $F : T' \to T''$ and an isomorphism of surfaces $H : \Omega' \to \Omega''$ such that $f'' \circ F = H \circ f'$. Note that if two G-coverings are equivalent then they are topologically equivalent. Converse statement is not generally true: there exist nonequivalent G-coverings f' and f'' of the same surface Ω which are topologically equivalent.

Certainly, all introduced notions can be reduced to 1-dimensional case, i.e. coverings of 1-dimensional stratified topological manifolds - circles, rays and segments - by 1-dimensional "seamed" spaces, i.e. seamed graphs. We skip the details.

2.3. **Local topological invariants of G-coverings.** Let $f : T \to \Omega$ be a G-covering, U be a small neighborhood of a point $p \in \Omega$. Denote by f_U the restriction of f to the preimage $\widehat{U} = f^{-1}(U)$. Clearly, group G acts on $\widehat{U}$ and $\widehat{U}/G \approx U$. Similar denotations we use for an arbitrary subset $U \subset \Omega$.

Fix a local orientation of Ω at point p. For points of any type we will describe complete set of invariants of the covering f_U up to topological equivalence preserving the orientations. We show below that coverings over a small neighborhood U can be reduced to coverings over 1-dimensional spaces: circles, rays and segments. In the case of one-dimensional coverings topological equivalence preserving orientations actually coincides with equivalence of coverings. In this sections topological equivalence means topological equivalence preserving local orientations.

Let p be a simple point of Ω. We may assume that the boundary $\partial U = \overline{U} \setminus U$ of U is a contour and all points of U and ∂U are simple. In that case the preimage $\widehat{U}$ of U consists of n discs, n is the degree of the covering. Each of discs is mapped onto U homeomorphically. Denote by K the stabilizer of a point $z \in D$ of a disc D that is a connected component of $\widehat{U}$. Group K acts trivially on D and $\widehat{U}$ is G-isomorphic to the cross-product $G \times_K D$. By definition, cross-product $G \times_K D$ is a set of equivalence classes in the direct product $G \times D$; the equivalence is defined by formula $(g, p) \sim (gk, k^{-1}p)$ for any $k \in K$. The action of group G on $G \times_K U$ is defined by the formula $g(h, p) = (gh, p)$.

G-covering $f_U : \widehat{U} \to U$ is topologically equivalent to the G-covering $G \times_K D \to D$. The stabilizer K depends on the choice of a disc D. Stabilizer K' of point z' from another disc is conjugated to K: $K' = gKg^{-1}$ where $g \in G$. Thus, the only local topological invariant of G-covering of Ω at simple points is a conjugacy class of the stabilizer K of any preimage of a simple point; $n = |G/K|$ is the degree of the covering. ($|M|$ denotes the cardinality of a set M.) Clearly, stabilizers of all simple points $z \in f^{-1}(\Omega^c)$ where Ω^c is a connected component of Ω are conjugated.

Let p be a marked interior point of Ω. We may assume that all points of U, except p, are simple and ∂U is homeomorphic to a circle. Therefore, G-covering $f_{\partial U} : \widehat{\partial U} \to \partial U$ is topologically equivalent to a G-covering of a circle. Clearly, $\widehat{\partial U}$ consists of a number of connected components, each one homeomorphic to a circle. Let S be one of the components. Fixed local orientation at point p induces the orientation of ∂U and S. Denote by K the stabilizer of a point $z \in S$. Clearly, stabilizers of all points of S coincide with K. A path around contour ∂U from $f(z)$ to $f(z)$ in the direction induced by the local

orientation at p induces the path in S from z to a point $\bar{z} \in f^{-1}(f(z))$ and, therefore, an element $a \in G$, such that $az = \bar{z}$. Element a belongs to the normalizer $N_G(K)$ and is defined up to the stabilizer K. Denote by K_a the subgroup of G generated by K and a. Group K_a acts on S with kernel K, and, clearly, $S/K_a \approx \partial U$. Therefore, G-covering $f_{\partial U} : \widehat{\partial U} \to \partial U$ is topologically equivalent to the G-covering $G \times_{K_a} S \to \partial U$.

If we take another connected component S' of $\widehat{\partial U}$ and another point $z' \in S'$ then we obtain another pair (K', a') where K' is the stabilizer of z' and a' is element of the normalizer $N_G(K')$. Clearly, there exists an element $g \in G$ such that $K' = gKg^{-1}$ and $a' = gag^{-1}$. Thus, the only topological invariant of G-covering over the boundary of U is a conjugacy class α of a pair (K, a) where K is a subgroup of G and a is an element of group $N = N_H(K)/K$.

Note that classes of topological equivalence of G-covering $f_{\partial U}$ of a boundary ∂U are in one-to-one correspondence with classes of topological equivalence of G-coverings f_U of U because U and $\widehat{U}$ can be reconstructed from ∂U and $\widehat{\partial U}$ by canonical operations: U is homeomorphic to an open cone over ∂U and any connected component of $\widehat{U}$ is homeomorphic to an open cone over a connected component of $\widehat{\partial U}$.

Denote by $\mathcal{A}$ the set of conjugacy classes of elements of group $N = N_G(K)/K$. It was shown that the topological invariant of a G-seamed covering at an interior marked point is an element of $\mathcal{A}$. An element of $\mathcal{A}$ we call 'an interior field' and denote usually by α.

Define an involute map $\mathcal{A} \to \mathcal{A}^*$ by formula $[a] \to [a]^* = [a^{-1}]$ where $[a]$ denotes the conjugacy class of element $a \in N$.

Elements of the centralizer $C_N(a)$ are in one-to-one correspondence with G-automorphisms of the G-covering $f_U : \widehat{U} \to U$. By this reason we define automorphism group of $\alpha \in \mathcal{A}$ as $\mathsf{Aut}\,\alpha = C_N(a)$ for an element $a \in \mathcal{A}$. The group $\mathsf{Aut}\,\alpha$ is defined up to isomorphism.

Let p belongs to an open 1*-stratum of* Ω. We may assume that small neighborhood U of p is homeomorphic to a rectangle $R = \{z \in \mathbb{C} || \operatorname{\mathsf{Re}} z| < 1, 0 \leq \operatorname{\mathsf{Im}} z < 1\}$, point p corresponds to 0 and $U \cap \partial\Omega$ corresponds to the segment $(-1, +1)$ of real axis. The preimage $\widehat{U}$ of U consists of a number of book-like seamed surfaces $B^1, B^2, \ldots$. Denote by R a ray $R \subset U$ that corresponds to the segment $[0, i)$ of imaginary axis. Then G-covering $f_R : \widehat{R} \to R$ is topologically equivalent to the G-covering of a ray. Choose a ray $\delta \subset \widehat{R}$. Denote by K common stabilizer of interior points of δ and by S the stabilizer of the vertex z of δ. Clearly, $K \subset S$. If we choose another ray instead of δ than we obtain conjugated pair (K', S') of subgroups: $K' = gKg^{-1}$, $S' = gSg^{-1}$ for some $g \in G$. Up to a topological equivalence G-covering of a ray is defined by conjugacy class of pair of subgroups (K, S). Indeed, group S acts on the connected component $\widehat{R}^c$ of $\widehat{R}$ that contains δ and $\widehat{R}^c/S \approx R$. $\widehat{R}^c$ consists of $|S/K|$ rays with common vertex. Therefore, G-covering $f_R : \widehat{R} \to R$ is topologically equivalent to the G-covering $G \times_S \widehat{R}^c$.

Classes of topological equivalence of G-covering f_R of ray R are in one-to-one correspondence with classes of topological equivalence of G-coverings f_U of U because U and $\widehat{U}$ can be reconstructed from R and $\widehat{R}$ by canonical operations: U is homeomorphic to an open cylinder over R and any connected component of $\widehat{U}$ is homeomorphic to an open cylinder over a connected component of $\widehat{R}$.

Thus, the only topological invariant at point p is a pair of subgroups (K, S) such that $K \subset S$ up to conjugation by element of G. Fix a subgroup K from the pair. Then the set of conjugacy classes of pairs is in one-to-one correspondence with classes of equivalence

of subgroups $S \subset G$ such that $S \supset K$; the equivalence is defined by $S \sim nSn^{-1}$ where $n \in N_G(K)$.

Let p be a boundary marked point. We may assume that the boundary $\partial U = \widehat{U} \setminus U$ of appropriate small neighborhood of p is homeomorphic to a closed segment and U is homeomorphic to an open cone over ∂U. Restriction $f_{\partial U}$ of G-covering f to ∂U is a G-covering of a segment. A connected component $\widehat{\partial U}^c$ of the preimage $\widehat{\partial U}$ is a connected bipartite graph. Choose an edge $e \in \widehat{\partial U}^c$ and denote vertexes of e by a and b. Vertexes are ordered according to the local orientation at p. Let K be common stabilizer of internal points of e, S_a be the stabilizer of a and S_b be the stabilizer of b. Clearly, $K \subset S_a \cap S_b$. Denote by S a subgroup generated by subgroups S_a and S_b. It can be shown that S acts on the component $\widehat{\partial U}^c$ and $\widehat{\partial U}^c/S \approx \partial U$. Moreover, the G-covering $f_{\partial U} : \widehat{\partial U} \to \partial U$ is topologically equivalent to the G-covering of cross-product $G \times_S \widehat{\partial U}^c$. Covering over ∂U defines the covering f_U of the neighborhood U up to topological equivalence because U is homeomorphic to an open cone over ∂U and $\widehat{U}$ is homeomorphic to the disjoint union of open cones over connected components of $\widehat{\partial U}$.

Thus, up to topological invariance, G-covering in the neighborhood of a marked boundary point is defined by an ordered triple of subgroups (S_a, K, S_b) such that $K \subset S_a \cap S_b$ up to equivalence $(S_a, K, S_b) \sim (gS_ag^{-1}, gKg^{-1}, gS_bg^{-1})$ for $g \in G$. Fix subgroup K from a triple. Clearly, the set of equivalent classes of triples (S_a, K, S_b) are in one-to-one correspondence with equivalent classes of pairs (S_a, S_b) such that $S_a \supset K$ and $S_b \supset K$; the equivalence is defined by $(S_a, K, S_b) \sim (nS_a, n^{-1}, nS_bn^{-1})$, $n \in N_G(K)$. We denote the latter set by $\mathcal{B}$. An element of $\mathcal{B}$ we call 'a boundary field' and denote usually by β.

We may assume that the group $N = N_G(K)/K$ acts on the set of subgroups S containing K by conjugations $S \to nSn^{-1}$, $n \in N$ because K acts trivially. Elements of N such that $nS_an^{-1} = S_a$, $nS_bn^{-1} = S_b$ are in one-to-one correspondence with G-automorphisms of the G-covering $f_U : \widehat{U} \to U$. By this reason we define automorphism group of $\beta \in \mathcal{B}$ as $\mathsf{Aut}\,\beta = \{n \in N | nS_an^{-1} = S_a, nS_bn^{-1} = S_b\}$ for a pair of subgroups $(S_a, S_b) \in \beta$. The group $\mathsf{Aut}\,\beta$ is defined up to isomorphism.

Define an involute map $\mathcal{B} \to \mathcal{B}$ by formula $[(S_a, S_b)] \to [(S_a, S_b)]^* = [(S_b, S_a)]$ where $[(S_a, S_b)]$ denotes the equivalence class of an ordered pair of subgroups.

Note that if Ω is connected surface and $f : T \to \Omega$ is a G-covering then stabilizers of all simple points of T are conjugated subgroups. In the case of nonconnected Ω we restrict ourselves to G-coverings such that stabilizers of simple points over all components of Ω are conjugated.

2.4. **Hurwitz numbers of G-coverings.** Fix a surface Ω, a finite group G and a subgroup K of G. Denote by Ω_a the set of interior marked points of Ω and by Ω_b the set of boundary marked points.

Let $f : T \to \Omega$ be a G-covering of Ω by G-seamed surface such that stabilizers of all simple points of T are conjugated to K. It was shown in section 2.3 that local topological invariants of f at interior marked points belong to the set $\mathcal{A}$ of conjugacy classes of the group $N_G(K)/K$ and local topological invariants of f at boundary marked points belong to the set $\mathcal{B}$ consisting of equivalence classes of ordered pairs (S_a, S_b) of subgroups such that $K \subset S_a \cap S_b$; the equivalence is defined as conjugation of a pair by an element $n \in N_G(K)$. Therefore, G-covering $f : T \to \Omega$ defines maps $\mathfrak{a}_f : \Omega_a \to \mathcal{A}$ and $\mathfrak{b}_f : \Omega_b \to \mathcal{B}$, the images $\mathfrak{a}_f(p)$ (resp., $\mathfrak{b}_f(q)$) we denote by α_p (resp.β_q) and call interior (resp., boundary) fields.

Fix arbitrary maps $\mathfrak{a} : \Omega_a \to \mathcal{A}$ and $\mathfrak{b} : \Omega_b \to \mathcal{B}$. Denote by $\mathsf{Cov}(\Omega, \mathfrak{a}, \mathfrak{b})$ the set of isomorphic classes of G-coverings of Ω by G-seamed surfaces such that its local invariant at any point $p \in \Omega_a$ is equal to $\mathfrak{a}(p)$ and its local invariants at any point $q \in \Omega_b$ is equal to $\mathfrak{b}(q)$.

The number $\mathcal{H}(\Omega; \mathfrak{a}; \mathfrak{b}) = \sum_{[f]\in \mathsf{Cov}(\Omega,\mathfrak{a},\mathfrak{b})} \frac{1}{|\mathsf{Aut}_G f|}$ where f is a representative of the isomorphic class $[f]$ of G-coverings is called *a Hurwitz number of G-coverings.*

There are relations between classical Hurwitz numbers for different Klein surfaces. Those relations correspond to cuts of surfaces. Similar relations we establish below for Hurwitz numbers of G-coverings by seamed surfaces.

Let Ω be a surface. A connected closed oriented curve $\gamma \subset \Omega$ without self-intersections is called *a simple cut* if it does not meet any marked point and is either a closed contour belonging to the interior of the surface or a segment such that $\partial\gamma = \gamma \cap \partial\Omega$.

Let $\gamma \subset \Omega$ be a simple cut. We will define a cut contracted surface $\Omega_\# = \Omega/\gamma$ in two steps. First, a cut surface Ω_* is a surface $(\Omega\setminus\gamma)\cup\widehat{\gamma}$ where $\widehat{\gamma}$ is the set of pairs (r, c), $r \in \gamma$, c is a co-orientation of γ at point r. It is clear how to define the topology on Ω_*, see [1] for the details. Clearly, there is a natural continuous map $\Omega_* \to \Omega$, its restriction to $\Omega_* \setminus \widehat{\gamma}$ is the homeomorphism with $\Omega \setminus \gamma$, its restriction to $\widehat{\gamma}$ is a double-covering of γ. Second, *a cut contracted surface* $\Omega_\#$ is a surface obtained by contracting each connected component of $\widehat{\gamma}$ into a point. Here $\widehat{\gamma}$ is the preimage of simple cut γ in cut surface Ω_*. Marked points of $\Omega_\#$ are marked points coming from Ω and points that are contracted components of $\widehat{\gamma}$. Note, that the orientation of γ induces local orientations at corresponding marked points.

Clearly, $\Omega_\#$ canonically corresponds to Ω despite of the absence of natural continuous map of Ω to $\Omega_\#$ (the homeomorphism between $\Omega \setminus \gamma$ and $\Omega_\# \setminus$ (image of $\widehat{\gamma}$) cannot be extended to γ).

Depending on a simple cut γ, three situations may occur.

(i) γ is co-orientable contour. Then $\widehat{\gamma}$ consists of two contours and $(\Omega_\#)_a = \Omega_a \sqcup \{p', p''\}$ where points $p^{(i)}$ correspond to components of $\widehat{\gamma}$; $(\Omega_\#)_b = \Omega_b$.

(ii) γ is non-coorientable contour. Then $\widehat{\gamma}$ consists of one contour and $(\Omega_\#)_a = \Omega_a \sqcup \{p'\}$ where p' corresponds to $\widehat{\gamma}$; $(\Omega_\#)_b = \Omega_b$.

(iii) γ is a segment. Then $\widehat{\gamma}$ consists of two segments and $(\Omega_\#)_b = \Omega_b \sqcup \{q', q''\}$ where points $q^{(i)}$ correspond to components of $\widehat{\gamma}$; $(\Omega_\#)_a = \Omega_a$.

Marked points p', p'' (in the case (i)), p (in the case (ii)) and q', q'' (in the case (iii)) we call 'extra marked points'.

Let $\mathfrak{a} : \Omega_a \to \mathcal{A}$ and $\mathfrak{b} : \Omega_b \to \mathcal{B}$ be two maps of sets of marked points of a surface Ω. We will extend them to obtain maps $\mathfrak{a}_{\Omega_\#} : (\Omega_\#)_a \to \mathcal{A}$ and $\mathfrak{b}_{\Omega_\#} : (\Omega_\#)_b \to \mathcal{B}$ for cut contracted surface $\Omega_\#$. The extended maps depend on an element $\alpha \in \mathcal{A}$ for simple cuts of types (i) and (ii) and on an element $\beta \in \mathcal{B}$ for a simple cut of type (iii). For marked points of $\Omega_\#$ coming from the marked points of Ω extended maps coincide with maps $\mathfrak{a}$ and $\mathfrak{b}$. Values of maps $\mathfrak{a}_{\Omega_\#}$ and $\mathfrak{b}_{\Omega_\#}$ at all extra marked points are equal to α (in cases (i), (ii)) and β (in case (iii)).

Identifying maps with sets of their images, we use below the following denotations.

(i) $\mathfrak{a}_{\Omega_\#} = (\mathfrak{a}, \alpha, \alpha)$

(ii) $\mathfrak{a}_{\Omega_\#} = (\mathfrak{a}, \alpha)$

(iii) $\mathfrak{b}_{\Omega_\#} = (\mathfrak{b}, \beta, \beta)$

The relations between Hurwitz numbers are described in the following important lemma.

Lemma 2.1. *Let $\gamma \subset \Omega$ be a simple cut of a surface Ω and $\Omega_\# = \Omega/\gamma$ be cut contracted surface.*

(i) If γ is a co-orientable contour, then

$$\mathcal{H}(\Omega; \mathfrak{a}; \mathfrak{b}) = \sum_{\alpha \in \mathcal{A}} |\operatorname{\mathsf{Aut}} \alpha| \mathcal{H}(\Omega_\#; \mathfrak{a}, \alpha, \alpha; \mathfrak{b})$$

where group $\mathsf{Aut}\,\alpha$ is defined in section 2.3.

(ii) If γ is a non-coorientable contour, then

$$\mathcal{H}(\Omega; \mathfrak{a}; \mathfrak{b}) = \sum_{\alpha \in \mathcal{A}} d^\alpha \mathcal{H}(\Omega_\#; \mathfrak{a}, \alpha; \mathfrak{b})$$

where d^α is the number of elements $n \in N = N_G(K)/K$ such that $n^2 = a^{-1}$ for a fixed representative $a \in \alpha$ of the conjugacy class α[1]*.*

(iii) If γ is a segment, then

$$\mathcal{H}(\Omega; \mathfrak{a}; \mathfrak{b}) = \sum_{\beta \in \mathcal{B}} |\operatorname{\mathsf{Aut}} \beta| \mathcal{H}(\Omega_\#; \mathfrak{a}; \mathfrak{b}, \beta, \beta)$$

where group $\mathsf{Aut}\,\beta$ is defined in section 2.3.

Actually, the proof follows the proof of similar lemma for coverings of surfaces by surfaces [1]. □

3. Klein Topological Field Theory for G-coverings of surfaces by seamed surfaces

We reproduce below the definition of Klein topological field theory given in [1].

3.1. **Category of surfaces and Klein functor.** In this section we define a category $\mathcal{S}$ of surfaces. An object of $\mathcal{S}$ is a surfaces Ω with marked points (see section 2.1) endowed with a set of local orientations at marked points. The local orientation at marked point $r \in \Omega_0$ is denoted by o_r.

We define morphisms of four types. First, an isomorphism of surfaces (see section 2.1) preserving local orientations at marked points is called a morphism in the category $\mathcal{S}$. Second, for an oriented cut $\gamma \subset \Omega$ the correspondence $\Omega \to \Omega_\#$ where $\Omega_\# = \Omega/\gamma$ is the cut contracted surface is called a morphism. Third, changing a local orientation at a marked point r to the opposite orientation is called a morphism. Fourth, marking off a new point and fixing a local orientation at it is called a morphism. An arbitrary morphism in $\mathcal{S}$ is defined as a combination of morphisms of those four types. The disjoint union $\Omega' \cup \Omega''$ of two surfaces we consider as a tensor product in $\mathcal{S}$; all axioms of tensor category are satisfied.

Let $\{X_m | m \in M\}$ be a finite set of $n = |M|$ vector spaces X_m over the field of complex numbers $\mathbb{C}$. The action of the symmetric group S_n on $\{1, \dots, n\}$ induces its action on the sum of the vector spaces $\left(\oplus_\sigma X_{\sigma(1)} \otimes \dots \otimes X_{\sigma(n)}\right)$ where σ runs through the bijections $\{1, \dots, n\} \to M$; an element $s \in S_n$ takes $X_{\sigma(1)} \otimes \dots \otimes X_{\sigma(n)}$ to $X_{\sigma(s(1))} \otimes \dots \otimes X_{\sigma(s(n))}$. Denote by $\otimes_{m \in M} X_m$ the subspace of all invariants of this action. The vector space $\otimes_{m \in M} X_m$ is canonical isomorphic to tensor product of all X_m in any fixed order; the isomorphism is the projection of $\otimes_{m \in M} X_m$ to the summand that is equal to the tensor product of X_m in that order.

[1]We correct here an incorrectness in the definition of numbers d^α in [1]

Assume that all X_m are equal to a fixed vector space X. Then any bijection $M \to M'$ of sets induces an isomorphism $\otimes_{m\in M} X \leftrightarrow \otimes_{m'\in M'} X$.

Fix a pair (A, B) of vector spaces over $\mathbb{C}$, a set of elements $1_A \in A$, $1_B \in B$, $K_A^{\otimes} \in A \otimes A$, $K_B^{\otimes} \in B \otimes B$, $U \in A$ and involute linear maps $\iota_A : A \to A$, $\iota_B : B \to B$ such that $\iota_A(1_A) = 1_A$, $\iota_B(1_B) = 1_B$. Denote by $\mathcal{N}$ the set of all those data: $\mathcal{N} = \{A, B, 1_A \in A, 1_B \in B, K_A^{\otimes} \in A \otimes A, K_B^{\otimes} \in B \otimes B, U \in A, \iota_A, \iota_B\}$. Using $\mathcal{N}$, we will define a functor $\mathcal{V}$ from the category $\mathcal{S}$ to the category of vector spaces.

Let Ω be an object of $\mathcal{S}$, Ω_0 be the set of marked points, $\Omega_a \subset \Omega_0$ be the set of interior marked points and $\Omega_b = \Omega_0 \setminus \Omega_a$ be the set of boundary marked points. Assign a copy A_p of vector space A to any $p \in \Omega_a$ and a copy B_q of vector space B to any $q \in \Omega_b$. Denote by V_Ω vector space $(\otimes_{p\in\Omega_a} A_p) \otimes (\otimes_{q\in\Omega_b} B_q)$ By definition, put $\mathcal{V}(\Omega) = V_\Omega$

Functor $\mathcal{V}$ takes morphisms of four types in the category $\mathcal{S}$ to the following morphisms of linear spaces. For any morphism ϕ in category $\mathcal{S}$ the morphism $\mathcal{V}(\phi)$ is denoted below by ϕ_*.

1) Let $\phi : \Omega' \to \Omega''$ be an isomorphism of surfaces. Then $\phi_* : V_{\Omega'} \to V_{\Omega''}$ is a linear map induced by the bijections $\phi|_{\Omega'_a} : \Omega'_a \to \Omega''_a$ and $\phi|_{\Omega'_b} : \Omega'_b \to \Omega''_b$.

2) Let $\psi : \Omega' \to \Omega''$ be a morphism of changing the local orientation at an interior point $r \in \Omega_0$. Then ψ_* is a liner map that is identical on all components of the tensor product $V_{\Omega'} = (\otimes_{p\in\Omega_a} A_p) \otimes (\otimes_{q\in\Omega_b} B_q)$ except A_r and it coincides with ι_A on A_r.

The morphism ψ_* for morphism ψ of changing the local orientation at a boundary marked point is defined similarly, just replace ι_A by ι_B.

3) Let $\xi : \Omega' \to \Omega''$ be a morphism of marking off an interior point $p \in \Omega' \setminus \Omega'_0$ and fixing a local orientation o_p. Vector space $V_{\Omega''}$ can be canonically identified with $V_{\Omega'} \otimes A$. Then $\xi_* : V_{\Omega'} \to V_{\Omega''}$ maps vector $x \in V_{\Omega'}$ to the vector of $V_{\Omega''}$ that corresponds to $x \otimes 1_A \in V_{\Omega'} \otimes A$. Property $\iota_A(1_A) = 1_A$ guaranties the correctness.

Similarly, the morphism ξ'_* corresponding to a morphism ξ' which is marking off a boundary point $q \in \partial\Omega' \setminus \Omega'_b$ takes vector $x \in V_{\Omega'}$ to the vector of $V_{\Omega''}$ corresponding to $(x \otimes 1_B) \in V_{\Omega'} \otimes B$.

4) Let $\eta : \Omega \to \Omega_\#$ be a morphism from Ω to cut contracted surface $\Omega_\#$ induced by an oriented cut $\gamma \subset \Omega$. As it was shown in section 2.4, η induces the inclusion map of the set $\Omega_0 = \Omega_a \cup \Omega_b$ of marked points of Ω into the set of marked points $\Omega_{\#0}$ of $\Omega_\#$ and $\Delta = \Omega_{\#0} \setminus \Omega_0$ consists of either one point or two points depending on γ.

If γ is non-coorientable contour, then Δ consist of one point, $V_{\Omega_\#}$ is canonically isomorphic to $V_\Omega \otimes A$ and $\eta_*(x) = x \otimes U$. If γ is a co-orientable contour then $|\Delta| = 2$, $V_{\Omega_\#} \equiv V_\Omega \otimes A \otimes A$ and $\eta_*(x) = x \otimes K_A^{\otimes}$. If γ is a segment then $|\Delta| = 2$, $V_{\Omega_\#} \equiv V_\Omega \otimes B \otimes B$ and $\eta_*(x) = x \otimes K_B^{\otimes}$.

In addition, the tensor product in $\mathcal{S}$ defined by the disjoint union of surfaces $\Omega' \otimes \Omega'' \to \Omega' \cup \Omega''$ induces the tensor product of vector spaces $\theta_* = V_{\Omega'} \otimes V_{\Omega''} \to V_{\Omega' \cup \Omega''}$.

The functorial properties of $\mathcal{V}$ can be easily verified.

3.2. **Klein Topological Field Theory.** Fix a set of data $\mathcal{N} = \{A, B, 1_A \in A, 1_B \in B, K_A^{\otimes} \in A \otimes A, K_B^{\otimes} \in B \otimes B, U \in A, \iota_A : A \to A, \iota_B : B \to B\}$ that defines a Klein functor $\mathcal{V}$.

A system of linear forms $\mathcal{F} = \{\Phi_\Omega : V_\Omega \to \mathbb{C}\}$, $\Omega \in \mathcal{S}$, is called a *Klein topological field theory (KTFT)* if it satisfies the following axioms.

1° *Topological invariance.*

$$\Phi_\Omega(\phi_*(x)) = \Phi_\Omega(x)$$

for any isomorphism $\phi : \Omega \to \Omega'$ of surfaces.

2° *Invariance under a change of local orientations.*

$$\Phi_{\Omega'}(\psi_*(x)) = \Phi_\Omega(x)$$

for any morphism $\psi : \Omega \to \Omega'$ of changing the local orientation at a marked point.

3° *Nondegeneracy.*

Define bilinear form $(x', x'')_A$ on vector space A by formula $(x', x'')_A = \Phi_{(S,p_1,p_2)}(x'_{p_1} \otimes x''_{p_2})$ where (S, p', p'') is a sphere with two marked points p' and p'', local orientations at p', p'' are induced by a global orientation of the sphere, $x'_{p_1} \otimes x''_{p_2}$ is the element of $V_{(S,p_1,p_2)}$ corresponding to $x' \otimes x'' \in A \otimes A$ under the canonical isomorphism $V_{(S,p_1,p_2)} \equiv A \otimes A$. It follows from axioms 1° and 2° that the definition is correct and bilinear form (x', x'') is symmetric. Taking a disc with two boundary marked points (D, q_1, q_2) instead of (S, p_1, p_2) we define a symmetric bilinear form $(x', x'')_B$ on vector space B by similar construction.

The axiom is:

both forms $(x', x'')_A$ and $(x', x'')_B$ are nondegenerate.

4° *Invariance under addition of marked point.*

$$\Phi_{\Omega'}(\xi_*(x)) = \Phi_\Omega(x)$$

for any morphism $\xi : \Omega \to \Omega'$ that is induced by marking off an additional point in Ω.

5° *Cut invariance.*

$$\Phi_{\Omega_\#}(\eta_*(x)) = \Phi_\Omega(x)$$

for any morphism $\eta : \Omega \to \Omega_\#$ of Ω to the cut contracted surface $\Omega_\#$ that is induced by an oriented simple cut $\gamma \subset \Omega$.

6° *Multiplicativity.*

$$\Phi_\Omega(\theta_*(x' \otimes x')) = \Phi_{\Omega'}(x')\Phi_{\Omega''}(x'')$$

for $\Omega = \Omega' \cup \Omega''$, $x' \in V_{\Omega'}$, $x'' \in V_{\Omega''}$ and isomorphism $\theta^* : V_{\Omega'} \otimes V_{\Omega''} \to V_\Omega$ corresponding to the tensor product $\theta : \Omega' \times \Omega'' \to \Omega$.

Lemma 3.1. *Let $\{\Phi_\Omega\}$ be a KTFT. Then the following relations hold:*
(i) $(1_A, \alpha)_A = \Phi_{S^2,p}(\alpha)$ where S^2 is a sphere with marked point p, a local orientation at p is induced by a global orientation of S^2, and $\alpha \in A$ is an arbitrary element;
(ii) $(1_B, \beta)_B = \Phi_{D^2,q}(\beta)$ where D^2 is a disc with one boundary marked point q, a local orientation at q is induced by a global orientation of D^2 and $\beta \in B$ is an arbitrary element;
(iii) $(U, \alpha)_A = \Phi_{P^2,p}(\alpha)$ where P^2 is a projective plane with marked point p and arbitrary local orientation at p;
(iv) $(\iota_A(\alpha'), \alpha'')_A = \Phi_{S^2,p',p''(-)}(\alpha', \alpha'')$ where S^2 is a sphere with marked points p' and p'', local orientation at p' coincides with a fixed global orientation of S^2 and local orientation at p'' does not coincide with the fixed global orientation of S^2;
(v) $(\iota_B(\beta'), \beta'')_B = \Phi_{D^2,q',q''(-)}(\beta', \beta'')$ where D^2 is a disc with marked points q' and q'',

local orientation at q' coincides with a fixed global orientation of D^2 and local orientation at q'' does not coincide with the fixed global orientation of D^2.
(vi) $(K_A^{\otimes}, \alpha' \otimes \alpha'')_A = (\alpha', \alpha'')_A$ *for any* $\alpha', \alpha'' \in A$;
(vii) $(K_B^{\otimes}, \beta' \otimes \beta'')_B = (\beta', \beta'')_B$ *for any* $\beta', \beta'' \in B$;

Proof. The proof is straightforward □

3.3. **Klein Topological Field Theory of G-coverings.** Let G be a finite group and K be a subgroup of G such that $\cap_{g\in G} gKg^{-1} = \{e\}$. We shell construct a KTFT corresponding to G-coverings of surfaces by seamed surfaces such that stabilizers of all simple points of cover seamed surface are conjugated to K.

Denote by $\mathcal{A}$ the set of all possible values of local invariants of G-coverings $T \to \Omega$ at interior marked points of the base. The set $\mathcal{A}$ can be identified with the set of conjugacy classes of elements of the factor-group $N = N_G(K)/K$ (see section 2.3). Denote by A a vector space of formal linear combinations of elements of $\mathcal{A}$.

Denote by $\mathcal{B}$ the set of all possible values of local invariants of G-coverings $T \to \Omega$ at boundary marked points of the base. The set $\mathcal{B}$ can be identified with the set of equivalence classes of ordered pairs of subgroups (S_1, S_2) of the group G such that $K \subset S_1 \cap S_2$; the equivalence is defined by formula $(S_1, S_2) \sim (gS_1g^{-1}, gS_2g^{-1})$ for any $g \in N_G(K)$ (see section 2.3). Denote by B a vector space of formal linear combinations of elements of $\mathcal{B}$.

For any surface Ω Hurwitz numbers $\mathcal{H}(\Omega; \mathfrak{a}, \mathfrak{b})$ where $\mathfrak{a} : \Omega_a \to \mathcal{A}$ and $\mathfrak{b} : \Omega_b \to \mathcal{B}$ are maps used in the definition of Hurwitz numbers, induce the linear function $\mathcal{H}_\Omega : V_\Omega \to \mathbb{C}$.

We will equip vector spaces A and B by a set of data needed to define a Klein functor (section 3.1).

By 1_A denote the conjugacy class of the unit in group $N = N_G(K)/K$; by definition, $1_A \in A$.

Let $\mathcal{R}$ be the set of equivalent classes of subgroups $S \subset G$ such that $S \supset K$; the equivalence is defined by formula $S \sim nSn^{-1}$ for any $n \in N_G(K)$. By 1_B denote the sum $\sum_{[S]\in\mathcal{S}}[(S,S)]$ where S is a representative of equivalence class $[S] \in \mathcal{R}$; by definition, $1_B \in B$.

Denote by $\iota_A : A \to A$ and $\iota_B : B \to B$ involute linear maps induced by involute maps $\alpha \to \alpha^*$ and $\beta \to \beta^*$ of $\mathcal{A}$ and $\mathcal{B}$ respectively (see section 2.3).

By $K_A^{\otimes}$ and $K_B^{\otimes}$ denote the elements $K_A^{\otimes} = \sum_{\alpha\in\mathcal{A}} |\mathsf{Aut}\,\alpha| \alpha \otimes \alpha^*$ and $K_B^{\otimes} = \sum_{\beta\in\mathcal{B}} |\mathsf{Aut}\,\beta| \beta \otimes \beta^*$ respectively; by definition, $K_A^{\otimes} \in A \otimes A$, $K_B^{\otimes} \in B \otimes B$.

Denote by U the element $U = \sum_{g\in N} g^2$ of A.

Thus, we defined the set of data
$\mathcal{N} = \{A, B, 1_A \in A, 1_B \in B, K_A^{\otimes} \in A \otimes A, K_B^{\otimes} \in B \otimes B, U \in A, \iota_A, \iota_B\}$, which is needed to construct a Klein functor (see section 3.1).

Theorem 3.1. *The system of linear functionals $\mathcal{H} = \{\mathcal{H}_\Omega\}$ is a Klein topological field theory. We call it topological field theory of G-coverings of surfaces by seamed surfaces.*

Proof. Let $T \to \Omega$ be a G-covering of Ω and $\alpha \in \Omega$ be a local invariant at interior marked point $p \in \Omega$. Note that local orientation at p is used to define α (see section 2.3). Changing the local orientation at p leads to replacing a conjugacy class $\alpha \subset N_G(K)/K$ by the conjugacy class by $\alpha^* = [a^{-1}]$ where $a \in \alpha$. That relation is equivalent to axiom 2° of KTFT for interior marked points. Axiom 2° for boundary marked points can be verified similarly.

Axiom 1° is a corollary of the topological invariance of Hurwitz numbers with respect to isomorphisms of surfaces preserving local orientations at marked points.

Axiom 3° (nondegeneracy of the form) follows from the straightforward calculations of Hurwitz numbers for a sphere with two interior marked points and a disc with two boundary marked points.

Axiom 4° is evidently satisfied for the marking off a new interior point p with $\mathfrak{a}(p) = 1_A$ because the conditions of general position are actually required from a covering at that point. The same is true for boundary marked points too.

Axiom 5° follows from lemma 2.1.

Axiom 6° follows from the definition of Hurwitz numbers.

□

4. Cardy-Frobenius algebras associated with actions of groups

It follows from [1] that KTFTs up to the equivalence are in one-to-one correspondence with a certain class of algebras up to isomorphisms. Algebras of that class was called 'structure algebras' in [1] and renamed to 'Cardy-Frobenius algebras' in [3]. We use the latter term here.

4.1. **Definition of Cardy-Frobenius algebra.** A Frobenius algebra D is an associative algebra such that there exists a nondegenerate symmetric invariant bilinear form $(x', x'')_D$ on D. The invariance means $(x'x'', x''')_D = (x', x''x''')_D$.

A finite dimensional Frobenius algebra D with unit $1_D \in D$ endowed with an involute anti-automorphism $d \mapsto d^*$ (i.e. $(x'x'')^* = x''^* x'^*$) and a linear form $l_D(x)$ such that the bilinear form $(x', x'')_D = l_D(x'x'')$ is symmetric, nondegenerate and $l_D(x^*) = l_D(x)$ is called *an equipped Frobenius algebra*.

Let D be an equipped Frobenius algebra. Choose a basis $\alpha_1, \ldots, \alpha_n$ of vector space D. Denote by $((F_{\alpha_i,\alpha_j}))$ the matrix of the bilinear form, $F_{\alpha_i,\alpha_j} = (\alpha_i, \alpha_j)_D$, by $((F^{\alpha_i,\alpha_j}))$ the matrix of the dual form and by $((F^*{}_{\alpha_i,\alpha_j}))$ (resp., $((F^{*\alpha_i,\alpha_j})))$ the matrix of twisted bilinear form $F^*{}_{\alpha_i,\alpha_j} = (\alpha_i, \alpha_j^*)_D$ (resp. dual to twisted bilinear form).

An element $K_D = F^{\alpha_i\alpha_j}\alpha_i\alpha_j$ of D is called *a Casimir element*. In that formula (and throughout the paper) the sum over repeated indexes is assumed. Define also *twisted Casimir element* $K_D^* \in D$ by formula $K_D^* = F^{*\alpha_i\alpha_j}\alpha_i\alpha_j$. It can be shown that Casimir element and twisted Casimir element don't depend on the choice of basis.

Fix a set of data $\mathcal{H} = \{A, B, \phi, U\}$, where A and B are equipped Frobenius algebras, A is a commutative algebra, $\phi : A \to B$ is a homomorphism of algebra A into the center of algebra B, $U \in A$ is an element. Denote by H an algebra structure on $A \oplus B$ defined by multiplications in A and B and by formula $ab = ba = \phi(a)b$ for $a \in A, b \in B$. Thus, A is a subalgebra and B is two-sided ideal of H.

The algebra $H = A \oplus B$ together with the set of data $\mathcal{H} = \{A, B, \phi, U\}$ is called *a Cardy-Frobenius algebra* if the following conditions hold:

(1) $\phi(x^*) = (\phi(x))^*$

(2) (Cardy condition) For any $x, y \in B$ denote by $W_{x,y} : B \to B$ a linear operator $W_{x,y}(z) = xzy$. Let $\phi^* : B \to A$ be the linear map dual to $\phi : A \to B$ (i.e. $(a, \phi^*(b))_A = (\phi(a), b)_B$, for $a \in A, b \in B$). It is required that for any $x, y \in B$ the following identity holds:

$$(\phi^*(x), \phi^*(y))_A = trW_{x,y}$$

(3) $U^2 = K_A^*$ and $\phi(U) = K_B^*$

4.2. **Semisimple Cardy-Frobenius algebras.** Direct sums of Cardy-Frobenius algebras, ideals etc. can be defined in usual way. We skip the details.

Example 4.1. *Let A be an algebra of complex numbers $\mathbb{C}$ equipped with identical involution and linear form $l_A(z) = \mu^2 z$ where $\mu \in \mathbb{C}$ is nonzero number. Then A is equipped Frobenius algebra.*

By definition, put $U = \frac{1}{\mu} \in A$.

Let B be a matrix algebra $\mathbb{M}(n, \mathbb{C})$ equipped with involute anti-automorphism $X \mapsto X'$, where X' is transposed matrix, and linear form $l_B(X) = \mu \operatorname{tr} X$. Then B is equipped Frobenius algebra. Define a homomorphism $\phi : A \to B$ by the equality $\phi(1) = E$ where $E \in B$ is the identity matrix.

It can be shown that the set of data $\{A, B, \phi, U\}$ defines the simple Cardy-Frobenius algebra $H = A \oplus B$. We call H a Cardy-Frobenius algebra $H^+_{(n,\mu)}$

Example 4.2. *Let A be equipped Frobenius algebra from previous example. By definition, put $U = -\frac{1}{\mu} \in A$.*

Let B be a matrix algebra $\mathbb{M}(2m, \mathbb{C})$ of even order. A matrix $X \in B$ we may present in block form as $X = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$. Define the involute anti-automorphism $X \mapsto X^\tau$ by formula $X^\tau = \begin{pmatrix} m'_{22} & -m'_{12} \\ -m'_{21} & m'_{11} \end{pmatrix}$, where m' denotes transposed matrix. Then algebra B equipped with involute anti-automorphism $X \mapsto X^\tau$ and linear form $l_B(X) = \mu \operatorname{tr} X$ is equipped Frobenius algebra.

Define homomorphism $\phi : A \to B$ by the equality $\phi(1) = E$ where $E \in B$ is the identity matrix.

It can be shown that the set of data $\{A, B, \phi, U\}$ defines the simple Cardy-Frobenius algebra $H = A \oplus B$. We call H a Cardy-Frobenius algebra $H^-_{(2m,\mu)}$

Example 4.3. *Let $A = \mathbb{C} \oplus \mathbb{C}$. Define an involution by the equality $(x, y)^* = (y, x)$ for $(x, y) \in \mathbb{C} \oplus \mathbb{C}$ and the linear form by formula $l_A(x, y) = \mu^2(x + y)$. Clearly, A is commutative equipped Frobenius algebra. Put $U = 0$.*

Let $B = \mathbb{M}(n, \mathbb{C}) \oplus \mathbb{M}(n, \mathbb{C})$ be the direct sum of two matrix algebras of equal order. Then algebra B equipped with the involute anti-automorphism $(X, Y) \mapsto (Y', X')$ and linear form

$$l_B(X, Y) = \mu(\operatorname{tr} X + \operatorname{tr} Y)$$

is an equipped Frobenius algebra.

Define a homomorphism $\phi : A \to B$ by the equality $\phi(x, y) = (xE, yE)$.

It can be shown that the set of data $\{A, B, \phi, U\}$ defines the simple Cardy-Frobenius algebra $H = A \oplus B$. We call H a Cardy-Frobenius algebra $H^0_{(2n,\mu)}$

Example 4.4. *Denote by A_μ the pair (A, U) from example 4.1. Denote by A^0_μ the pair (A, U) from example 4.3. Note that algebras $A_\mu \oplus \{0\}$ and $A^0_\mu \oplus \{0\}$ satisfy all conditions of Cardy-Frobenius algebras except one: B has no unit.*

It was proven [1], that any Cardy-Frobenius algebra such that both algebras A and B are semisimple is isomorphic to the direct sum of simple Cardy-Frobenius algebras of types $H^+_{(n,\mu)}$, $H^-_{(2m,\mu)}$, $H^0_{(2n,\mu)}$ and commutative equipped Frobenius algebras with fixed $U \in A$ of types A_μ and A^0_μ.

4.3. **Cardy-Frobenius algebras associated with actions of groups.** We will construct here a wide class of Cardy-Frobenius algebras $H = A \oplus B$. Among algebras B (noncommutative parts of Cardy-Frobenius algebras) there are, for example, all Hecke algebras of finite groups [21]. In the next section we will show that one of Cardy-Frobenius algebras of that class coincides with Cardy-Frobenius algebra of G-coverings of surfaces by seamed surfaces.

Let N be a finite group. Group algebra $\mathbb{C}[N]$ can be equipped with involute antiautomorphism $x \mapsto x^*$ defined on $g \in N$ by formula $g^* = g^{-1}$ and linear form $l(x)$ defined by formula $l(g) = \delta_{g,e}$ for $g \in N$ (e denotes the unit of N).

Denote by A the center of the group algebra . Clearly, A is invariant under the involution $x \mapsto x^*$. It can be easily shown that the algebra A endowed with that involution and the restriction of linear form $l(x)$ is commutative equipped Frobenius algebra.

Denote by U the element $U = \sum_{g\in N} g^2$ of group algebra. Clearly, $U \in A$.

Lemma 4.1. *The square of element U is equal to the twisted Cazimir element: $U^2 = K_A^*$*

Proof. We have $U^2 = \sum_{g,h\in N} g^2h^2$.

Denote by $\mathcal{A}$ the set of conjugacy classes of the group N. For a conjugacy class α put $E_\alpha = \sum_{g\in\alpha} g \in \mathbb{C}[N]$. It is known that elements E_α belong to the center A of the group algebra and form a basis of A.

We have $K_A^* = \sum_{\alpha\in\mathcal{A}} |\operatorname{Aut}\alpha| E_\alpha^2 = \sum_{\alpha\in\mathcal{A}} |\operatorname{Aut}\alpha| (\sum_{g\in\alpha} g)^2 = \sum_{g,h\in N} ghg^{-1}h$. Using substitutions $a = gh$ and $b = h^{-1}g^{-1}h$ we get $\sum_{g,h\in N} ghg^{-1}h = \sum_{a,b\in N} a^2b^2$. Therefore, $U^2 = K_A^*$ $\square$

Suppose, finite group N acts effectively on a finite set X. Define by X^n the direct product of n copies of X. Group N acts on X^n by formula $g(x_1,\dots,x_n) = (g(x_1),\dots,g(x_n))$ where $x_1,\dots,x_n \in X$, $g \in N$. Denote by $\mathcal{B}_n$ the set of orbits X^n/N.

Denote by $\operatorname{Aut}\bar{x}$ the stabilizer of element $\bar{x} = (x_1,\dots,x_n) \in X^n$ Clearly, $\operatorname{Aut}\bar{x} = \cap_i \operatorname{Aut} x_i$.

Let B be the vector space of formal linear combinations with complex coefficients of elements of $\mathcal{B}_2$.

Fix an ordered set $(\beta_1,\dots,\beta_n)$ of elements $\beta_i \in \mathcal{B}_2$. By definition, an element β_i is an orbit of the action of group N on X^2. Denote by $\mathcal{B}_n(\beta_1,\dots,\beta_n)$ the set of orbits $[\xi]$ in X^n such that the following condition holds: for a representative $\xi = (x_1,\dots,x_n) \in [\xi]$ the pair (x_1,x_2) belongs to the orbit β_1, (x_2,x_3) belongs to β_2, $\dots$, (x_{n-1},x_n) belongs to β_{n-1}, (x_n,x_1) belongs to β_n.

Define a number $T_{\beta_1,\dots,\beta_n}$ by formula

$$T_{\beta_1,\dots,\beta_n} = \sum_{[\xi]\in\mathcal{B}_n(\beta_1,\dots,\beta_n)} \frac{1}{\operatorname{Aut}\xi},$$

where ξ is a representative of an orbit $[\xi] \in \mathcal{B}_n(\beta_1,\dots,\beta_n)$.

Numbers $\{T_{\beta_1,\dots,\beta_n}\}$ can be considered as a tensor, it corresponds to a polylinear form $B^{\otimes n} \to \mathbb{C}$.

The correspondence $(x_1,x_2) \mapsto (x_2,x_1)$ generates an involute linear transformation of B which we denote by $\beta \mapsto \beta^*$.

Tensor T_{β_1,β_2} defines a bilinear form on B, which we denote by $(\beta',\beta'')_B$. By direct calculations we obtain $(\beta_1,\beta_2)_B = \delta_{\beta_1,\beta_2^*}\frac{1}{|\operatorname{Aut}\beta_1|}$. Therefore, bilinear form $(\beta',\beta'')_B$ is symmetric and non-degenerate.

Using the bilinear form $(\beta', \beta'')_B$ we may convert tensor $T_{\beta_1,\beta_2,\beta_3}$ to a multiplication on B. Indeed, formula $(\beta_1 \cdot \beta_2, \beta_3) = T_{\beta_1,\beta_2,\beta_3}$ correctly defines an element $\beta_1 \cdot \beta_2$. The obtained algebra is denoted by the same letter B.

Denote by $\mathcal{B}_2^circ$ the set of orbits of elements $(x, x) \in X^2$, $x \in X$. By straightforward computations we get that the element $1_B = \sum_{\beta \in \mathcal{B}_2^\circ} \beta$ is a unit of algebra B. Formula $l_B(\beta) = (\beta, 1_B)_B$ defines a linear form.

Lemma 4.2. *Algebra B endowed with linear form l_B and involution $\beta \mapsto \beta^*$ is an equipped Frobenius algebra.*

The proof is by direct calculations □

Denote by V_X the vector space of formal linear combinations with complex coefficients of elements of set X and extend the action of N on X to the representation ρ of N in V_X.

For any $\beta \in \mathcal{B}_2$ define the operator $V_\beta \in \mathsf{End}\, V_X$ by formula $V_\beta = \sum_{(x_1,x_2)\in\beta} E_{x_1,x_2}$. Here E_{x_1,x_2} is the elementary matrix: $E_{x_1,x_2} x = \delta_{x_2,x} x_1$ for any $x \in X$. Denote by ν the linear map $B \to \mathsf{End}\, V_X$, induced by the correspondence $\beta \to V_\beta$.

Lemma 4.3. *The map $\nu : B \to \mathsf{End}\, V_X$ is a faithful representation of algebra B. The image $\nu(B) \subset \mathsf{End}\, V_X$ coincides with the centralizer of the image $\rho(N) \subset \mathsf{End}\, V_X$ of group N under the representation $\rho : N \to \mathsf{End}\, V_X$.*

Proof. Let $((M_{x,y}))$ be the matrix of a linear transformation $M : V_X \to V_X$ in the basis, consisting of elements $x \in X$. Then by direct calculations we obtain that F commutes with $\rho(N)$ if and only if matrix elements $M_{x,y}$ and $M_{g(x),g(y)}$ are equal for any $(x, y) \in X^2$ and $g \in N$. This claim is equivalent to the claim of lemma □

Corollary 4.1. *Algebra B is semisimple equipped Frobenius algebra.*

Proof. Indeed, the centralizer of a semisimple subalgebra of semisimple algebra is semisimple □

The constructed equipped Frobenius algebra B depends on group N and its action on finite set X. We denote that algebra by $B_{N:X}$. Algebra A acts on V_X because N acts on V_X. Clearly, that representation of A is faithful. By lemma 4.3, the image $\rho(A)$ is contained in the center of the image of algebra B. The representation of B in V_X is faithful, therefore, we obtain the homomorphism of A into B. That homomorphism we denote by ϕ.

Theorem 4.1. *Let a finite group N acts effectively on a set X. Then defined above objects: the algebras A, B, the homomorphism $\phi : A \to B$ and the element $U \in A$, generate a Cardy-Frobenius algebra $H = A \oplus B$.*

Proof. Clearly, that involute anti-automorphism $\beta \mapsto \beta^*$ acts on the image of B in $\mathsf{End}\, V_X$ as the transposition $M \mapsto M'$ of matrices. Denote by V_α the image of element $E_\alpha \in A$ in $\mathsf{End}\, V_X$ under the representation $\rho : \mathbb{C}[N] \to \mathsf{End}\, V_X$. Obviously, $V'_\alpha = V_{\alpha^*}$. Therefore, condition (2) of the definition of Cardy-Frobenius algebra is satisfied.

The proof of Cardy relation is literally the same as the proof of that relation in [1] for special class of Cardy-Frobenius algebras, namely, Cardy-Frobenius algebras associated with finite groups. The proof is rather technical, we outline here main steps.

Compute first right side expression $R = \mathsf{tr}\, W_{x,y}$.

(1) Scalar product of elements $\beta_i \in \mathcal{B}_2$ can be computed in the representation of algebra B in $\mathsf{End}\, V_X$ as follows: $(\beta_1, \beta_2)_B = \frac{1}{|G|} \mathsf{tr}(V_{\beta_1} V_{\beta_2})$ where V_{β_i} are operators defined above.

(2) By direct calculations using exact formulas for matrix elements of V_β we get $R = \sum_\beta \frac{|\operatorname{Aut}\beta|}{|G|} \operatorname{tr}(V_\beta V_{\beta_1} V_{\beta^*} V_{\beta_2})$.

(3) Multiplying matrices in (2) we obtain

$R = \frac{1}{|G|}|P|$, where P is the set of tuples (x_1, x_2, x_3, x_4, g), $x_i \in X$, $g \in G$ such that $(x_1, x_2) \in \beta_1$, $(x_3, x_4) \in \beta_2$ and $g(x_2) = x_1$, $g(x_3) = x_4$.

Compute left side expression $L = (\phi^*(\beta_1), \phi^*(\beta_2))_A$.

For any $\alpha \in \mathcal{A}$ we have $V_\alpha = \sum_{x\in X, g\in\alpha} E_{g(x),x}$, where E_{x_1,x_2} is an elementary matrix. (Recall that $V_\alpha \in \operatorname{End} V_X$ is the image of element $E_\alpha \in A$ in $\operatorname{End} V_X$ under the representation $\rho : \mathbb{C}[G] \to \operatorname{End} V_X$).

(4) By direct calculations we obtain $L = \sum_\alpha |\operatorname{Aut}\alpha| \frac{1}{|G|^2} \operatorname{tr}(V_\alpha V_{\beta_1}) \operatorname{tr}(V_{\alpha^*} V_{\beta_2})$.

(5) The equality (4) can be rewritten in the form

$L = \sum_\alpha |\operatorname{Aut}\alpha| \frac{1}{|G|^2}|Q|$ where Q is the set of tuples $(x_1, x_2, x_3, x_4, a_1, a_2)$, $x_i \in X$, $a_j \in \alpha$ such that $(x_1, x_2) \in \beta_1$, $(x_3, x_4) \in \beta_2$ and $a_1(x_1) = x_2$, $a_2(x_4) = x_3$.

(6) Representing element a_2 in the form ga_1g^{-1} we can rewrite (5) as follows:

$L = \sum_\alpha \frac{1}{|G|^2}|Q'|$ where Q' is the set of tuples $(x_1, x_2, x_3, x_4, a, g)$, $x_i \in X$, $a \in \alpha$, $g \in G$ such that $(x_1, x_2) \in \beta_1$, $(x_3, x_4) \in \beta_2$ and $a(x_1) = x_2$, $gag^{-1}(x_4) = x_3$.

(7) The equality $gag^{-1}(x_4) = x_3$ can be rewritten as $a(x_4') = x_3'$ where $x_i' = g^{-1}(x_i)$. For two fixed $g = g_1$ and $g = g_2$ the number of tuples $(x_1, x_2, x_3, x_4, a, g_1) \in Q'$ is equal to the number of tuples $(x_1, x_2, x_3, x_4, a, g_2) \in Q'$ since the pairs (x_3, x_4) run throughout all representatives of class β_2. Therefore, $L = \frac{1}{|G|}|P|$. Thus, we prove $L = R$.

To complete the proof, it is necessary to check that $\rho(U) = \nu(K_B^*)$. By definition, $U = \sum_{g\in N} g^2$. Therefore, $\rho(U) = \sum_{g\in N} E_{g^2(x),x}$. Denote $\nu(K_B^*)$ by K. By direct calculations we we obtain $K = \sum_\beta |\operatorname{Aut}\beta| \sum_{(x_1,x_2)\in\beta} E_{x_1,x_2} \sum_{(y_1,y_2)\in\beta} E_{y_1,y_2}$. We can substitute (y_1, y_2) by $(g(x_1), g(x_2)$ for some element $g \in G$ because both pairs (x_1, x_2) and (y_1, y_2) belong to the same class β. Taking in account the substitution and multiplying matrices we get $K = \sum_{x\in X, g\in G} E_{x,g^2(x)}$. Thus, $K = \rho(U)$ □

We denote constructed Cardy-Frobenius algebra by $H_{G:X}$.

Example 4.5. *Let symmetric group S_n acts on a finite set X of n elements. Then algebra $B_{S_n:X}$ is isomorphic to the algebra of bipartite graphs* [2, 3]*. Moreover, $H_{S_n:X}$ coincides with Cardy-Frobenius algebra of n-coverings of Klein surfaces by seamed surfaces* [2, 3].

Example 4.6. *Let X be the set of involutions in a finite group G. Group G acts on X by conjugations. Then $H_{G:X}$ coincides with constructed in* [1] *Cardy-Frobenius algebra associated with G. For symmetric group $G = S_n$ the latter algebra coincides with Cardy-Frobenius algebra of coverings of degree n for real algebraic curves* [1].

Example 4.7. *Let S be a subgroup of finite group G and $X = G/S$. Then algebra $B_{G:X}$ is isomorphic to Hecke algebra. Indeed, pairs of cosets $(xB, yB) \in X \times X$ are in one-to-one correspondence with double-cosets BgB and formulas for the multiplication in algebra $B_{G:X}$ coincide with formulas for the multiplication of double-cosets in Hecke algebra* [21].

5. Formula for Hurwitz numbers of G-coverings of surfaces by seamed surfaces

5.1. **Cardy-Frobenius algebra corresponding to a Klein topological field theory.** Klein topological field theories (KTFTs) are in one-to-one correspondence with

Cardy-Frobenius algebras [1]. Following [1] we briefly outline the construction of Cardy-Frobenius algebra corresponding to a KTFT.

By definition, Cardy-Frobenius algebra is generated by the following set of objects: a commutative equipped Frobenius algebra A, an equipped Frobenius algebra B, a homomorphism $\phi : A \to B$, an element $U \in A$.

Let $\mathcal{F}$ be a KTFT. By definition, $\mathcal{F}$ is a system of linear forms $\{\Phi_\Omega : V_\Omega \to \mathbb{C}\}$, defined for each surface Ω and satisfying certain axioms (see section 3.2). Vector spaces V_Ω are defined by the use of fixed Klein functor $\mathcal{V}$. In turn, Klein functor corresponds to a set of data $\mathcal{N}$; $\mathcal{N}$ includes, among others, two vector spaces A and B, elements $1_A \in A$, $U \in A$, $1_B \in B$, involute linear maps $\iota_A : A \to A$, $\iota_B : B \to B$ (see section 3.1). Using Klein functor $\mathcal{V}$ and KTFT $\mathcal{F}$, one can define algebra structures on vector spaces A and B and equip those algebras with required in the definition of Cardy-Frobenius algebras additional objects.

Let Ω be a sphere with two marked points, $\Omega = (S^2, p', p'')$. Then corresponding to Ω linear form $\Phi_{(S^2,p',p'')} : A \otimes A \to \mathbb{C}$ induces bilinear form $(x, y)_A$ on A. By axioms, the form is symmetric and nondegenerate. Similarly, disc (D^2, q', q'') with to boundary marked points give rise to nondegenerate bilinear form on B.

By definition, tri-linear map $\Phi : A \otimes A \otimes A \to \mathbb{C}$ corresponds to a sphere with three marked points. Using bilinear form on A, a tri-linear form can be converted into the multiplication $A \otimes A \to A$. Axioms of KTFT provide the commutativity and associativity of that multiplication and the invariance of the bilinear form. By construction, algebra A is equipped with the involute map ι_A, fixed element 1_A and linear form $l_A(x) = (x, 1_A)_A$. It can be shown that A is commutative equipped Frobenius algebra.

Similarly, a triangle, i.e. a disk with three marked boundary points, gives rise to algebra structure on B. It can be proven that B is (typically, noncommutative) equipped Frobenius algebra.

Let $\Omega = (D^2, p, q)$ be a disc with one boundary and one interior marked points. Then linear form $\Phi_{(D^2,p,q)} : A \otimes B \to \mathbb{C}$ corresponds to Ω. Using bilinear forms, the map $\Phi_{(D^2,p,q)}$ can be transformed into the map $\phi : A \to B$. It can be shown that ϕ is a homomorphism of algebras.

In [1] was proven that the data $\mathcal{H} = \{A, B, \phi, U\}$ correctly defines the structure of Cardy-Frobenius algebra on $H = A \oplus B$. In the proof different systems of simple cuts for certain surfaces were considered. For example, two essentially different systems of simple cuts of a cylinder with two marked boundary points belonging to different connected components of the boundary were used to prove Cardy relation.

5.2. **Cardy-Frobenius algebra of G-coverings of surfaces by seamed surfaces.** Let G be a finite group, $K \subset G$ be a subgroup such that $\cap_{g\in G} gKg^{-1} = \{e\}$. In section 3.3 KTFT $\mathcal{H}$ of G-coverings with stabilizers of simple points conjugated to K of surfaces by seamed surfaces was associated with the pair $G \supset K$. Denote by H the Cardy-Frobenius algebra corresponding to KTFT $\mathcal{H}$.

Denote by N the factor group $N_G(K)/K$ and by X the set of all subgroups $S \subset G$ such that $S \supset K$. Conjugations $S \to hSh^{-1}$, $h \in N_G(K)$ generate the action of N on X because elements of K acts trivially. Denote by $H_{N:X}$ Cardy-Frobenius algebra associated with that action (see section 4.3).

Theorem 5.1. *Cardy-Frobenius algebra $H_{N:X}$ is isomorphic to Cardy-Frobenius algebra H of G-coverings with stabilizers of simple points conjugated to K of surfaces by seamed surfaces.*

Proof. Both algebras H and $H_{N:X}$ were defined by choosing bases of vector spaces and explicit formulas for the multiplications of elements of the bases. Comparing those data for H (see section 5.1) and $H_{N:X}$ (see section 4.3) we observe that bases of both Cardy-Frobenius algebras are in one-to-one correspondence and formulas for the multiplications coincide. Thus, the algebras are isomorphic. We skip the details □

Example 5.1. *Let G be the smallest finite simple group A_5 and K be a subgroup of two elements, $K = Z_2$. Denote by $H = A \oplus B$ the Cardy-Frobenius algebra of the KTFT of A_5-coverings with stabilizers of simple points conjugated to Z_2 of surfaces by seamed surfaces. Clearly, subgroup $N_G(K)$ is isomorphic to $Z_2 \times Z_2$, therefore, the group $N = N_G(K)/K$ consists of two elements.*

We shell computed H using the isomorphism $H \approx H_{N:X}$ where X is the set of all subgroups of A_5 containing K.

By the above, algebra A is isomorphic to the center of group algebra $\mathbb{C}[N]$. Therefore, $A = \mathbb{C} \oplus \mathbb{C}$.

We claim that $B = M(1, \mathbb{C}) \oplus M(5, \mathbb{C})$. Indeed, there is six subgroups containing K: (1) $K = Z_2$, (2) $D_4 = Z_2 \times Z_2$, (3) A_4, (4) $G = A_5$ and (5-6) two conjugated in G subgroups of order 10; those subgroups are denoted by L'_{10} and L''_{10}, each of them is semidirect product of Z_2 and Z_5 with non-trivial action of Z_2 on Z_5. Thus, the set X consists of six elements. Non-unit element n of group N acts of X by permutation of L'_{10} and L''_{10}, other elements of X are fixed points of n.

Denote by V_X the vector space of linear combinations with complex coefficients of elements of X. Clearly, the centralizer of the image of N in $\operatorname{End} V_X$ is isomorphic to $M(1, \mathbb{C}) \oplus M(5, \mathbb{C})$. By lemma 4.3 B is isomorphic to that centralizer. Homomorphism $\phi : A \to B$ is actually an isomorphism of A with the center of algebra B.

Equipments of algebras A and B can be easily computed by lemma 4.2. Thus, Cardy-Frobenius algebra H is described.

Note, that G-coverings with certain restrictions on local topological invariants at marked points can be described by a similar approach. Indeed, let G be a finite group, K be its subgroup and Σ be a set of subgroups such that for any $S \in \Sigma$ (1) $S \supset K$ and (2) $gSg^{-1} \in \Sigma$ for any $g \in N_G(K)$. Then group $N = N_G(K)/K$ acts on Σ. Denote by $H_{N:\Sigma}$ a Cardy-Frobenius algebra corresponding to that action. Clearly, $H_{N:\Sigma}$ is a subalgebra of Cardy-Frobenius algebra $H_{N:X}$ where X is the set of all subgroups containing K. The subalgebra $H_{N:\Sigma}$can be considered as Cardy-Frobenius algebra of G-coverings of surfaces by seamed surfaces with restricted sets of local invariants.

Example 5.2. *Let G be symmetric group S_n, $K = \{e\}$ and Σ the set of subgroups of order 2 and 1. Then $N = N_G(\{e\})$ coincides with G. It can be shown that Cardy-Frobenius algebra $H_{S_n:\Sigma}$ is isomorphic to Cardy-Frobenius algebra of real algebraic curves constructed in* [1].

5.3. **Formula for Hurwitz numbers.** Let G be a finite group, $K \subset G$ be a subgroup and H be Cardy-Frobenius algebra corresponding to KTFT of G-coverings of surfaces by seamed surfaces such that all stabilizers of simple points of cover space are conjugated to K. Following [1], we will express Hurwitz numbers $\mathcal{H}(\Omega, \alpha, \beta)$ in terms of algebra H.

First, we may restrict ourselves to connected surfaces Ω due to multiplicativity of Hurwitz numbers with respect to the disjoint union of surfaces.

Second, a connected surface is defined, up to isomorphism, by the following topological invariants:

- orientability ε ($\varepsilon = 1$ for orientable surfaces, $\varepsilon = 0$ for non-orientable surfaces;
- genus g; for non-orientable surfaces g is half-integer: $g = \frac{1}{2}$ for projective plan, $g = 1$ for Klein bottle etc.
- number s of connected components of the boundary $\partial\Omega$;
- number m of interior marked points;
- numbers $m_1, \dots, m_s$ of marked points at each boundary contour.

If Ω is orientable we choose local orientations at all marked points compatible with a fixed orientation of Ω. If Ω is non-orientable then we choose arbitrary local orientations at interior boundary points and for each boundary contour we choose local orientations at boundary marked points lying at that contour compatible with fixed orientation of the contour.

Third, for given map $\alpha : \Omega_a \to \mathcal{A}$ denote by $\alpha_1, \dots \alpha_m$ images of α, i.e. interior fields. For given map $\beta : \Omega_b \to \mathcal{B}$ denote by $\beta^i_1, \dots, \beta^i_{m_i}$ images of marked points lying at i-th boundary contour ordered according the orientation of that boundary contour.

Fourth, according the definition, the structure of Cardy-Frobenius algebra H includes commutative equipped Frobenius algebra A, equipped Frobenius algebra B, homomorphism $\phi : A \to B$ and element $U \in A$. Linear forms $l_A : A \to \mathbb{C}$ and $l_B : B \to \mathbb{C}$ are those that used in the definition of equipped Frobenius algebras (see section 4.1).

Denote by $K_A \in A$ and $K_B \in B$ Cazimir elements of A and B respectively. K^g_A is just g-th power of the element in algebra A. All those elements for the algebra H were described in section 4.3.

Theorem 5.2. *(i) Let Ω be an orientable surface ($\varepsilon = 1$). Then we have:*

$$\mathcal{H}(\Omega, \{\alpha_p\}, \{\beta_q\}) = l_B(\phi(\alpha_1 ... \alpha_m K^g_A)\beta^1_1 \dots \beta^1_{m_1} K_B \beta^2_1 \dots \beta^2_{m_2} \dots K_B \beta^s_1 \dots \beta^s_{m_s})$$

(ii) Let Ω be a non-orientable marked surface. Then we have:

$$\mathcal{H}(\Omega, \{\alpha_p\}, \{\beta_q\}) = l_B(\phi(\alpha_1 ..... \alpha_m U^{2g})\beta^1_1 \dots \beta^1_{m_1} K_B \beta^2_1 \dots \beta^2_{m_2} \dots K_B \beta^s_1 ... \beta^s_{m_s})$$

Proof. The theorem follows from [1], theorem 4.4. □

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A.N.Belozersky Institute, Moscow State University, Moscow 119899,
E-mail address: aba@belozersky.msu.ru

A.N.Belozersky Institute, Moscow State University, Moscow 119899, Independent University of Moscow, Institute Theoretical and Experimental Physics
E-mail address: natanzon@mccme.ru